THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-Office at Springfield, Missouri, as Second-Class Mail Matter.

VOL. II. NOVEMBER, 1895. No. 11.

BIOGRAPHY

ORMOND STONE.

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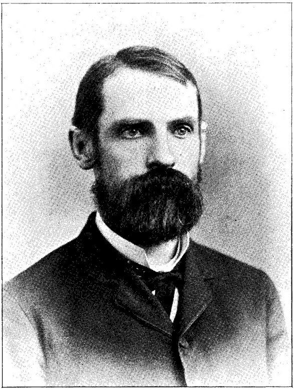
ORMOND STONE was born January 11, 1847, at Pekin, Tazewell County, Illinois, and was the oldest son of Rev. E. Stone, a travelling preacher of the Illinois Conference of the Methodist Episcopal Church. His father was of New England origin; his mother, Scotch-Irish.

In those early days, Illinois was on the frontier; and the Methodist preachers had large circuits which they changed every year or two. Among the places in which he lived were Canton, Nauvoo, and Carthage. From the last of which places the family moved in 1853 to Cook County, and have remained since then in Northern Illinois.

The boy showed a love for mathematics, when a mere child. At the age of seven, while living in the village of Libertyville, he discovered a copy of a new arithmetic, Adams's, which an attempt had been made to conceal from him. This he read twice, working all the problems each time, in a space of less than six weeks.

The next year his father moved to DeKalb Center, where his interest in mathematics was further advanced by an acquaintance with Dr. Matteson, a notice of whom has already appeared in this journal. A few years afterwards his father was stationed in Chicago, where the youth passed through the public schools of the city.

While still in the High School, the Dearborn Observatory was founded in connection with the old University of Chicago, whither Professor Safford



ORMOND STONE.

was called and remained in charge until the great fire in 1872. Young Stone soon made his acquaintance and almost immediately became his pupil, and thus began his career as an astronomer.

After graduating at the High School, he taught one year at Racine College; after which he returned to Chicago to continue his studies at the University. In 1869, in company with Professor Safford, he went to Des Moines, Iowa, to observe the great eclipse of that year. While there, he made the acquaintance of the astronomers sent from the Washington Observatory; and as a result, the next spring, he became an *assistant* in that institution. He was assigned to the Meridian Circle, on which he was employed for the next five years.

In 1875, he was called to the *Directorship* of the Cincinnati Observatory. Here, in connection with his assistants, he employed the 11-inch Equatorial of that institution in an extended and practically complete series of measures of the then known southern double-stars north of 30° south declination. Here, also, he commenced his work as *a trainer of young astronomers*, of whom now probably a larger number occupy important astronomical positions than the pupils of any other teacher in America.

In 1882, he was invited *to take charge of the new Leander McCormick Observatory of the University of Virginia.* This had not then been built. The great 26-inch telescope was finally ready for use in the spring of 1885. This building is memorable as possessing the first large dome made by Warner and Swasey. For the *first* time, also, in this country, electricity was applied to the illumination of the circles and micrometer of the great Refractor.

As the southern *double-stars* had been observed at Cincinnati, it was appropriate that he should devote this larger instrument to observations of southern *nebulæ*. As a result, hundreds of new nebulæ were discovered; and in 1893 there was published a catalogue of the micrometric measurements of the positions of southern nebulæ,—the only extended series of such measurements ever made in this country.

Meanwhile Professor Stone has made a special study of the great nebula of Orion, including a great number of photometric observations of the *condensations* of the Huyghenian region, and of the stars, especially of the *variables*, contained therein.

On the completion of the tenth volume of the *Analyst*, published by the late Dr. Hendricks, of Des Moines, Iowa, when that journal ceased to exist, Professor Stone began the publication of the *Annals of Mathematics*. For a time the editorship was shared with him by his colleague, Professor William M. Thornton; but at the close of the second volume, Professor Stone took entire charge, and the journal has been in his hands ever since.

In this elegantly printed bi-monthly journal, some very select problems are proposed for solution; and the solutions of the problems proposed are published as soon as possible. The *main object* of the publication of the *Annals of Mathematics*, by Professor Stone, is *to encourage mathematical research.*

Professor Stone is a brother of Mr. Melville E. Stone, of Chicago, the well-

known journalist, the founder of the Chicago *Daily News*, and the general manager of the Associated Press.

Professor Stone has written various papers on mathematical and astronomical subjects, which have appeared from time to time in the *Astronomische Nachrichten*, in *Gould's Astronomical Journal*, and in the *Annals of Mathematics*.

Professor Stone is also a member of a number of learned societies. In 1888 he was Chairman of the section of Mathematics and Astronomy of the American Association for the Advancement of Science ; and he is at present a member of the Council of the American Mathematical Society.

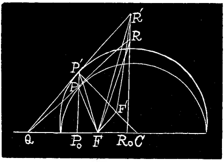
AN ELEMENTARY DERIVATION OF THE LAW OF GRAVITATION AS APPLIED TO PLANETARY MOTIONS.

By ORMOND STONE, University of Virginia.

The following derivation of the law of gravitation from Kepler's first two laws of planetary motion without the use of the machinery of the infinitesimal calculus is a modification of that given by Moebius. The loss by fire of a large portion of the library of the University of Virginia prevents my giving the place in his works where it may be found. As given by Moebius a slight knowledge of solid geometry is required ; as here given all the operations are performed in the plane of the orbit. The mass of the planet has been neglected.

Draw a circle having the major axis of the orbit as a diameter. Assume a point P' having such a motion that it is always at the intersection of the circumference of this circle and a straight line drawn through the planet P perpendicular to the major axis of the planet's orbit. The components of the velocities of P and P' in the direction parallel to the major axis are thus equal.

Draw QR tangent to the ellipse at P, and QR' tangent to the circle at P'. Q is situated on the major axis extended. If $PR=V$ represent the velocity of P and $P'R'=V'$ represent the velocity of P', RR' will be parallel to PP'. Let P_0 and R_0 be the intersections of PP' and RR' with the major axis of the orbit. Then by a property of the ellipse



$$P_0P=P_0P' \cos \varphi,\ R_0R=R_0R' \cos \varphi,$$

in which φ is the angle whose sine is e, the eccentricity of the orbit.

By one of Kepler's laws the sun is at F, the focus of the ellipse. PRF represents the areal velocity of P, and $P'R'F$ the areal velocity of P' with reference to F. As is easily seen,

$$PRF = P'R'F \cos \varphi\,;$$

whence, since by one of Kepler's laws PRF is constant, $P'R'F = c'$ is also a constant, and the acceleration of P' is directed toward F (see Young's General Astronomy, Art. 406).

Let A and A' be the total accelerations of P and P', and A_0 and A_0' be the components of these accelerations parallel to the major axis of the ellipse. Evidently

$$\frac{A_0}{A} = \frac{P_0F}{PF}, \qquad \frac{A_0'}{A'} = \frac{P_0F}{P'F};$$

whence, since $A_0 = A_0'$,

$$\frac{A}{A'} = \frac{P'F}{PF}. \tag{1}$$

Let C be the center of the ellipse, and F' the foot of the perpendicular from F on $P'C$. The component of A' in the direction $P'C$ is

$$A' \cos FP'F' = A'\frac{F'P'}{FP'} = \frac{V'^2}{a} \tag{2}$$

(see Young's General Astronomy, Art. 411).

Put $\angle FCP' = E =$ eccentric anomaly, and $FP = r =$ radius vector. We have also

$$\begin{aligned} CF &= ae, \\ CF' &= ae \cos E, \\ F'P' = CP' - CF' &= a(1 - e \cos E), \\ P_0P &= a \cos \varphi \sin E, \\ FP_0 &= a\,(\cos E - e). \end{aligned}$$

The last two equations give

$$FP = \sqrt{\overline{FP_0}^2 + \overline{P_0P}^2} = a(1 - e \cos E);$$

whence

$$F'P' = FP = r,$$

and (1) and (2) give

$$A = A'\frac{F'P'}{FP'} = \frac{V'^2}{a}. \tag{3}$$

Since FF'' is parallel to $P'R'$, the area of $P'F''R'$ is equal to that of $P'FR'$, which has already been shown to be constant; whence

$$F''P' \times P'R' = rV' = 2c', \text{ or } V' = \frac{2c'}{r}.$$

Substituting this in (3), we have

$$A = \frac{4c'^2}{a} \cdot \frac{1}{r^2}. \qquad \text{Q. E. D.}$$

A NOTE ON MEAN VALUES.

By E. H. MOORE, Ph. D., Professor of Mathematics in the University of Chicago.

A problem in averages or mean values usually reads thus:

(*A*) *Given a certain totality* $\Omega[\psi]$ *of objects* ψ, *and a certain function* $f(\psi)$ *of every object* ψ; *required the mean value* f_Ω *of the* $f(\psi)$ *for the* ψ*'s of the totality* $\Omega[\psi]$.

If the totality $\Omega[\psi]$ contains a *finite* number n of objects ψ — $\psi_1, \psi_2 \dots \psi_n$ — then we have the formula

$$(1) \qquad f_\Omega = \frac{\sum_{i=1}^{n} f(\psi_i)}{n}.$$

If the totality $\Omega[\psi]$ does not contain a finite number of objects, then the problem as stated (*A*) is *indefinite*. [The solution (1) cannot be directly generalized. To say that the number n is ∞ means merely that the totality $\Omega[\psi]$ is without number, that there is *no* such *number* n.]

To make (*A*) *definite we must supplement it by an explicit statement of a law of distribution of the objects* ψ *over the totality* $\Omega[\psi]$. In the ordinary cases this law of distribution makes ψ depend uniquely upon certain m independent variables $u_1, u_2 \dots u_m$, write it $\psi = \psi(u_1 \dots u_m)$, in such a way that the totality $\Omega[\psi]$ defines a certain totality $\bar{\Omega}[u_1 \dots u_m]$, and the function $f(\psi)$ becomes $f(\psi) = \bar{f}(u_1 \dots u_m)$. Now if the m-ple definite integrals

$$(2) \qquad I_1 = \int \dots \int \bar{f}(u_1 \dots u_m)\, du_1 \dots du_m, \quad I_2 = \int \dots \int du_1 \dots du_m$$

taken over the totality $\bar{\Omega}[u_1 \dots u_m]$ have definite meaning, (whether *or not* the

integral can be explicitly evaluated in terms of the commoner functions) then as the mean f_Ω of $f(\psi)$ for this distribution of ψ over $\Omega[\psi]$ we have

$$f_\Omega=\frac{I_1}{I_2}. \tag{3}$$

As has appeared time after time in all periodicals having a department devoted to mean values, when a problem is proposed in the indefinite form (A), different solvers introduce different laws of distribution, [each one the law appearing to him the most natural, or often enough, the law for which he can explicitly evaluate the integrals (2)]. These laws of distribution should be introduced explicitly as needed to make the problem definite. The possible laws of distribution are without number.

There is *no such thing* as THE *correct* solution of a problem stated in the indefinite form (A).

I make these quite obvious remarks for the sake of those persons who enjoy plunging into the whirl of the integrations without due meditation on the essential nature of the problem they are attacking.

The University of Chicago, November 5, 1895.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from September-October Number.]

SYSTEMS OF INTRANSITIVITY 3, 3.

(a) By multiplying every substitution of one group by every substitution of the other group we obtain

1. A group of order 36, viz:

$$\begin{array}{r|l} 1 & 1^* \\ abc & def \\ acb & dfe \\ ab & de \\ ac & df \\ bc & ef \end{array}$$

*This notation indicates that every substitution on one side of the line is multiplied into every one on the other side.

2. A group of order 18, viz:

$$\begin{array}{r|l} 1 & 1 \\ abc & def \\ acb & dfe \\ & de \\ & df \\ & ef \end{array}$$

3. A group of order 9, viz:

$$\begin{array}{r|l} 1 & 1 \\ abc & def \\ acb & dfe \end{array}$$

(*b*) By making the identical substitutions in the two systems correspond we obtain

4. A group of order 6, viz:

$$\begin{array}{c} 1 \\ abc.def \\ acb.dfe \\ ab.de \\ ac.df \\ bc.ef \end{array}$$

5. A group of order 3, viz:

$$\begin{array}{c} 1 \\ abc.def \\ acb.dfe \end{array}$$

(*c*) By multiplying the divisions according to a self-conjugate subgroup of one group into the corresponding divisions of the other we obtain

6. A group of order 18, viz:

$$\begin{array}{r|l} 1 & 1\,* \\ abc & def \\ acb & dfe \\ \hline ab & de \\ ac & df \\ bc & ef \end{array}$$

*This notation implies that the divisions of one column are to be multiplied into the corresponding divisions of the other column, Thus, *ab* is multiplied into *de*, *df*, and *ef*.

Systems of Intransitivity 2, 4.

There are five transitive groups of degree four, viz :

$$(abcd)all,\ (abcd)pos,\ (abcd)_8,\ (abcd)_4,\ (abcd)cyc$$

$$(abcd)_8=\begin{array}{llll} 1 & ac & ab.cd & abcd \\ & bd & ac.bd & adcb \\ & & ad.bc & \end{array}$$

$$(abcd)_4=\begin{array}{ll} 1 & ab.cd \\ & ac.bd \\ & ad.bc \end{array} \qquad (abcd)cyc=\begin{array}{lll} 1 & ac.bd & abcd \\ & & adcb \end{array}$$

By multiplying every substitution in each of these groups by $1,ef$ we clearly obtain five additional intransitive groups of degree six whose orders are 48, 24, 16, 8, and 8 respectively. It remains to find the groups which can be obtained by multiplying the self conjugate subgroups which include half the substitutions of these groups by 1 and the remaining substitutions by ef.

Since $(abcd)all$ and $(abcd)cyc$ have one such self conjugate subgroup apiece, they give rise to two groups of the required type. $(abcd)_8$ and $(abcd)_4$ have three such subgroups apiecc, but in the latter case they are all conjugate, as we have already seen ; hence we obtain the following four groups from these two groups :

$$\begin{array}{cccc}
(1) & (2) & (3) & (4) \\
\begin{array}{r|l} 1 & \\ ac & \\ bd & 1 \\ ac.bd & \\ \hline ab.cd & \\ ad.bc & ef \\ abcd & \\ adcb & \end{array} &
\begin{array}{r|l} 1 & \\ ab.cd & \\ ac.bd & 1 \\ ad.bc & \\ \hline ac & \\ bd & ef \\ abcd & \\ adcb & \end{array} &
\begin{array}{l|l} 1 & \\ ac.bd & \\ abcd & 1 \\ adcb & \\ \hline ac & \\ bd & ef \\ ab.cd & \\ ad.bc & \end{array} &
\begin{array}{r|l} 1 & \\ ab.cd & 1 \\ \hline ac.bd & ef \\ ad.bc & \end{array}
\end{array}$$

We have thus found that there are eleven intransitive groups of degree six whose systems of intransitivity are 2 and 4.

Systems of Intransitivity 2, 2, 2.

If we restrict ourselves to the first two systems we obtain an intransitive group of degree four. Our problem is thus reduced to the finding all the intransitive groups that can be obtained by combining one of the two intransitive groups of degree four with the group $1,ef$.

By multiplying every substitution of these two groups into $1,ef$ we obtain two groups whose degrees are four and eight respectively. By multiplying one

into the self conjugate subgroups of these groups and ef into the remaining substitutions we obtain the following additional groups :

$$\begin{array}{c|c} 1 & 1 \\ \hline ac.bd & ef \end{array} \qquad \text{and} \qquad \begin{array}{c|c} 1 & \\ ac.bd & 1 \\ \hline ac & ef \\ bd & \end{array}$$

We have now worked over the entire field and found twenty-one intransitive groups of degree six. These may be written as follows :*

LIST OF INTRANSITIVE GROUPS OF DEGREE SIX.

Order	No.	Group
2	1	$(ac.bd.ef)$
3	1	$(abc.def)$cyc
4	1	$(ab.cd)(ef)$
	2	$\{(ab)(cd)(ef)\}$pos
	3	$\{(abcd)\text{cyc}(ef)\}$pos
	4	$\{(abcd)_4(ef)\}$dim
6	1	$(abc.def)$all
8	1	$(ab)(cd)(ef)$
	2	$(abcd)\text{cyc}(ef)$
	3	$(abcd)_4(ef)$
	4—6	$\{(abcd)_8(ef)\}$
9	1	$(abc)\text{cyc}(def)$cyc
16	1	$(abcd)_8(ef)$
18	1	$(abc)\text{all}(def)$cyc
	2	$\{(abc)\text{all}(def)\text{all}\}$pos
24	1	$\{(abcd)\text{all}(ef)\}$pos
	2	$(abcd)\text{pos}(ef)$
36	1	$(abc)\text{all}(def)$all
48	1	$(abcd)\text{all}(ef)$

From what has been said it may be seen that intransitive groups of a given degree n can be readily constructed provided the transitive groups degree k are known, where $k=2, 3, 4, \ldots . n-2$. When n does not exceed 9 it is not difficult to write down all the possible intransitive groups, but for larger values of n the number of groups is so large that the construction of all such groups becomes comparatively quite laborious. The transitive groups seem much more important than the intransitive ones and we proceed to give methods by which they may be constructed.

Definition.—If a transitive group is such that any β letters may be replaced

*Cf. Professor Cayley : *Quarterly Journal of Mathematics*, vol. 25, pp. 71—79.

by any required set of β letters taken in any one of the $\beta!$ possible ways it is said to be β-fold transitive. If $\beta=1$ the group is one-fold or *simply* transitive.

Definition.—A transitive group whose letters can be divided into systems such that all the substitutions of the group perform upon these systems only one or both of the operations (1) interchanging the systems as units and (2) permuting the letters in the systems, is called a *non-primitive group.** All other transitive groups are called *primitive.*

We shall consider separately the construction of these two classes of transitive groups.

CONSTRUCTION OF THE NON-PRIMITIVE GROUPS.

Since the non-primitive group must be transitive it is always possible to replace any system by any required system by means of some substitution of the group. Hence it follows that the same number of letters is found in each system.

If $G=s_1, s_2, \ldots, s_g$ is any non-primitive group and $G_1=s_1, s_2, \ldots, s_{g_1}$ are the substitutions of G which do not interchange any of the systems then will G_1 be an *intransitive self conjugate subgroup of G.*

G_1 is a group because the substitutions

$$s_\alpha s_\beta \quad \begin{matrix} \alpha=1, 2, \ldots g_1 \\ \beta=1, 2, \ldots {}_2g_1 \end{matrix}$$

cannot interchange any of the systems. They must therefore be found in G_1 since they certainly belong to G.

G_1 is a self conjugate subgroup of G because

$$s_\gamma^{-1} G_1 s_\gamma \quad g_1<\gamma<g+1$$

must be a subgroup of G, which does not interchange the systems (the interchange effected by s_γ^{-1} being restored by s_γ) and must therefore be G_1 itself.

We may suppose that the transitive constituents of G_1 form the systems. For if $a_1, a_2, \ldots, a_e$ and $b_1, b_2, \ldots, b_k$ are transitively connected by G_1 then G contains some substitution s_γ which replaces a_1 by b_1. We proved above that

$$s_\gamma^{-1} s_\alpha s_\gamma = s_\beta \quad \begin{matrix} \alpha=1,2,\ldots, g_1 \\ \beta=1,2,\ldots, g_1 \end{matrix}$$

Suppose s_α so chosen that it replaces a_1 by a_2. Then it follows that s_γ replaces a_2 by one of the given b's. Similarly, we may show that s_γ replaces all of the given a's by the given b's. Since the a's and b's can be interchanged

*Such a group must be *simply* transitive, for if $\beta>1$ it would be possible to replace two letters of the same system by letters of different systems.

in this course of reasoning we have that $e=k$, or, more generally speaking, that

$$s_\gamma \qquad g<\gamma<g+1$$

either does not interchange any elements of two transitive constituents of G_1 or it interchanges all. Hence the transitive constituents satify the definition of systems.*

We are now prepared to see that the construction of non-primitive groups consists of two operations :

1. The construction of an intransitive group G_1 whose transitive constituents are conjugate groups.†

2. The construction of the substitutions of G which interchange the systems.

The first of these operations is a special case under the construction of intransitive groups and needs therefore no further attention. With respect to the second we shall first consider some special cases and then take up the general problem.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins), Member of the London Mathematical Society, and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from September-October Number.]

SCHOLION II, *in which is weighed the idea of that brilliant man Giovanni Alfonso Borelli in his Euclides Restitutus.*

This most learned author blames Euclid, because he defines parallel straight lines to be those, *which lying in the same plane do not meet on either side, even if produced into the infinite.*

He offers as ground for his accusation, that such relation is unknown: *first*, he says, *because we are ignorant whether such infinite non-concurrent lines can be found in nature; then also because we cannot perceive the properties of the infinite, and hence a relation of this sort is not clearly cognized.*

But with reverence for so great a man it may be said: can Euclid be blamed, because (to bring forward one among innumerable examples) he de-

*Cf. Netto's Theory of Substitutions (Cole's edition), §67.

†It has not been proved that all groups of this form can be used for G_1 but that every G_1 is of this form. The former cannot be proved.

fines *a square* to be *a figure quadrilateral, equilateral, rectangular;* when it may be doubted, whether a figure of this sort has place in nature ? He could, say I, most justly have been blamed, if, before as a Problem demonstrating the construction, he had assumed the aforesaid figure as given.

But that Euclid is free from this fault follows manifestly from this, that he nowhere presumes a square by itself explained, except after Proposition 46 of the First Book, in which in form of a problem he teaches, and demonstrates *the description from a given straight line, of the square as defined by him.*

In the same way therefore Euclid ought not to be blamed, because he defined parallel straight lines in this manner, since he nowhere assumes them as given for the construction of any problem, except after Proposition 31 of book first, in which as a Problem he demonstrates, how *should be drawn from a given point without a given straight line a straight line parallel to this,* and indeed according to the definition of parallels given by him, *so that produced indeed into the infinite on neither side do they meet one another.* And what is more; he demonstrates this without any dependence from the Postulate here controverted. And so Euclid demonstrates without any petitio principii that *there can be found in nature two such straight lines, which* (lying in the same plane) *protracted on each side into the infinite never meet,* and therefore *makes clearly known to us that relation,* by which he defines parallel straight lines.

Let us continue onward, whither the scrupulous accuser of Euclid invites us. Parallel straight lines he calls any two straights AC, BD, which stand at right angles to one certain straight AB (fig. with me 21). I admit, that a definition of this sort is set forth *by a state* (as he says) possible and most evident; since (Eu. I, 11.) from any point in the given straight a perpendicular can be erected.

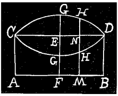


Fig. 21.

But precisely this both possibility and clearness I have just now demonstrated about the definition propounded by Euclid.

Wherefore remains only to compare that known Postulate of Euclid with this the other like postulate, which must be used for farther progress after new definition of parallels.

But behold this other postulate in Clavius (to whom Borelli himself expressly refers) in the Scholion after Proposition 18, book first: If a straight line, as suppose BD upon another straight, as suppose BA, moves transversely making with it at its extremity B always right angles, its other extremity D describes a line also straight, until this BD shall have come to congruence with the other equal AC. I acknowledge the fitness of the postulate, that thence a transit may be made to demonstrating that other Euclidean, upon which certainly at length must be supported all remaining geometry. For Clavius had previously declared; that a line, of which all points are equally distant from a certain assumed straight AB; as assuredly is (from the hypothesis of the aforesaid construction) the line DC; this line also must be straight; because certainly it will

be of this sort, that all its intermediate points *lie ex aequo* (such is the definition of a straight line) between its extreme points D, and C; *lie ex aequo*, say I, since all are equally distant from this assumed straight AB, truly as much as is the length of this BD, or AC. In this place Clavius introduces the example of the circular line, of which we will speak more conveniently below; where I will show the clearest disparity in this part between the straight line and circle.

For meanwhile I say it is not sufficiently evident, whether the line described by this point D is rather the straight DC than a certain curve DGC either convex or concave toward the side of this BA.

For if from the point F bisecting this BA a perpendicular is supposed erected, which meets the straight DC in E, and the aforesaid curves in G, and G, it follows surely (from P. II.) that the angles at the point E will be right; whatever line DC is understood at length as described in this motion by the point D; and moreover (from an easily understood superposition) the angles at the points G will be equal according as the one or the other curve DGC may be described.

But again; any point M in AB being assumed; if a perpendicular is erected, which meets the straight DC in N, and the aforesaid curves in H and H, I will prove a little later that the angles on both sides at the point N will be right, is as far indeed as this straight DC is supposed generated by the point D in that motion of its, or in as far as the straight MN is decided equal to this BD.

But if one or the other curve DHC is supposed generated; from the like aforesaid easy superposition will be demonstated that again the angles MHD, MHC on both sides will be equal, wherever in the one or the other described curve the point H may be assumed, from which to the straight line AB lying under the perpendicular HM is understood as let fall. But of this thing more fully and more scrupulously in the other part of this book, where it has its proper place.

To what end therefore, you will say, this untimely anticipation?

To this end, say I; lest from this indubitable property of the line generated in this manner, proved by me most rigorously in the aforesaid place; and indeed beyond any defect of any sort infinitely small; we may decide precipatately that the line can be only the straight.

Obviously the nature of the straight line must here be investigated more profoundly, without which geometry scarcely grown beyond infancy must there remain. Therefore in this affair cannot be blamed a greater investigation of a certain most exact verity.

Nor yet do I here deny, but that by certain most accurate physical experimentation can be discovered, that the line DC generated by this motion might be determined not other than a straight line.

But in so far as it may be here allowable to cite physical experimentation, I may forthwith bring forward three demonstrations physico-geometric to sanction the Euclidean Postulate.

Therewith I do not speak of physical experimentation extending into the

infinite, and therefore impossible for us; such as of course would be required to the cognizing, that all points of the straight join DC are equidistant from the straight AB, which is supposed to be in the some plane with this DC.

For a single individual case will be sufficient for me; as suppose, if, the straight DC being joined, and any one point of it N being assumed, the perpendicular NM let fall to the underlying AB is ascertained to be equal to BD or AC. For then the angles on both sides at the point N would be equal (P. I.) to the angles corresponding to them at the points C and D, which again (from the same P. I.) would be equal *inter se*. Wherefore the angles on both sides at the point N, and therefore also the remaining two will be right.

Therefore we will have a case for the hypothesis of right angle; and consequently (by PP. V. and XIII.) we will have demonstrated the Euclidean Postulate. And this may be the first demonstration physico-geometric.

I pass over to the second. Let there be a semicircle, of which the center is D, and diameter AC. If then (fig. 17) any point B is assumed in its circumference, to which AB, CB joined are ascertained to contain a right angle, this single case will be sufficient (as I have demonstrated in P. XVIII.) for establishing the hypothesis of right angle, and consequently (from the aforesaid P. XIII.) for demonstrating that well known Postulate.

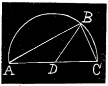


Fig. 17.

There remains the third demonstration physico-geometric, which I think the most efficacious, and most simple of all, inasmuch as it rests upon an accessible, most easy, and most convenient experiment.

For if in a circle, whose center is D, are fitted (fig. 22) three straight lines CB, BL, LA, each equal to the radius DC, and it is ascertained that the join AC goes through the center D, this will be sufficient for demonstrating the assertion.

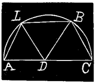


Fig. 22.

For, DB, DL being joined, we will have three triangles, which (from Eu. I. 8 and 5) not only will be equiangular to one another, but also singly for themselves. Therefore since the three angles together at the point D, indeed ADL, LDB, BDC are equal (by Eu. I. 13) to two right angles; also the three angles together of each of these triangles will be equal to two right angles, as suppose of the triangle BDC. Wherefore (from P. XV.) will be established hence the hypothesis of right angle; and consequently (from the already used P. XIII.) that Postulate will be demonstrated.

But if, before all attempt whether at demonstration or at graphic exhibition, one wishes to compare *inter se* those two postulates, I grant indeed the Euclidean may appear more obscure or even liable to objection. But after the graphic exhibition, which I reserve for Scholium IV. following, it will appear *vice versa* that the Euclidean Postulate indeed can retain the dignity and name of postulate, but the other ought rather to be reckoned among the theorems.

But here I must explain (as a little above I have promised I was about to

do) the manifest disparity in this relation between the circular line and the straight line. Now the disparity arises from this; that a line is called straight in reference to itself; but is called circular, as suppose (fig. 23) *MDHNM*, not in reference to itself, but to something else, forsooth to a certain other point *A* existing in the same plane with it, which is its center.

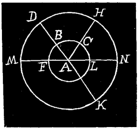


Fig. 23.

The consequence therefore is, as is demonstrated most excellently by Clavius, that the line *FBCL* existing in the same plane with it, and whose points are all equidistant from the aforesaid *MDHNM*, is also itself circular, truly equidistant in all its points from the common center *A*. That in fact *BD*, which is the continuation in a straight of *AB*, is the measure of the distance of that point *B* from this circle *MDHNM* follows from this; because (from Eu. III. 7, which is independent of the postulate here in controversy) this is the smallest of all, which can fall from this point upon this circumference. The same holds of the remaining *CH*, *LN*, *FM*.

Since therefore also the wholes *AM*, *AD*, *AH* are equal as radii from the center *A* to the line assumed circular *MDHNM*; and also the sections *FM*, *BD*, *CH*, *LN* are equal, which obviously are the measure of the equal distance of all points of that line *FBCLF* from this line presumed circular *MDHNM*; the consequence plainly is, that equal likewise are the remainders *AF*, *AB*, *AC*, *AL*, and therefore also this line is a circle with the same center *A*.

But now likewise, for demonstrating, that the line *DC* (fig. 21) generated through such a motion by the point *D* is a straight line will the equidistance of all its points from the underlying straight *BA* be sufficient? In no way.

For a line is called straight absolutely in reference to itself, or in itself, doubtless as *lying ex aequo between its points*, and especially end points, so that these remaining unmoved it cannot be revolved into occupying a new place. Unless this state in some way be demonstrated of this *DC* it will never be certain that this is a straight line, whatever relation finally is supposed or demonstrated of all its points to the underlying straight *AB* in the same plane; but especially we must not say analogically that no other line in this plane will be straight which in all its points is not equidistant from this line *AB* supposed straight.

Nor finally do I wish this dictum of mine so taken, as if I think it cannot be demonstrated, that the line thus generated is itself a straight line, except after truth demonstrated of the controverted postulate; since rather I myself will demonstrate it toward the end of this book, for confirming the like postulate itself.

[To be Continued.]

ASTRONOMICAL DETERMINATION OF THE TIME OF THALES.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

According to Pliny, the *cosmical setting* of the Pleiades, during the time of Thales of Miletus, occurred twenty-five days after the Vernal Equinox.

The geographical latitude of Miletus is $\lambda=+37^\circ\ 30'$. The brightest star of the Pleiades is η *Tauri*, popularly known as *Alcyone*. The apparent place for upper transit of η Tauri, at Washington, for the Mean Solar Date, January 9.3, 1895, is, according to Professor Simon Newcomb, characterized by a *Right Ascension*, $\alpha=3h.\ 41m.\ 15.49s.$, and by a *Declination*, $\delta=+23^\circ\ 47'\ 0''.4$.

From these data the hour-angle of η Tauri is determined to be $h=109^\circ\ 45'\ 54''.6, =7h.\ 19m.\ 3.64s.$ Hence the *sidereal time* of the *rising* of η Tauri is $T_r=20h.\ 22m.\ 11.85s.$, and *that* of the *setting* is $T_s=11h.\ 0m.\ 19.13s.$; that is, η Tauri is $T=14h.\ 38m.\ 7.28s.$ of sidereal time above the horizon.

This problem can now be solved in four different ways; and possibly the most expeditious method of *mathematical* solution, is to make $T_s=(180^\circ-\Theta)=165^\circ\ 4'\ 46''.95$. Therefore $\Theta=14^\circ\ 55'\ 13''.05$.

By a calculation analogous to that in our second solution of the "Dog-Star Problem," we find $\frac{1}{2}(Z+V)=87^\circ\ 27'\ 17''.81$, and $\frac{1}{2}(Z-V)=68^\circ\ 54'\ 20''.94$. Therefore, $Z=156^\circ\ 21'\ 38''.75$, and $V=18^\circ\ 32'\ 56'.87$; and $VZ=39^\circ\ 57'\ 0''.032$.

The value of the obliquity of the ecliptic, as furnished us by Professor Young, of Princeton, is $\omega=23^\circ\ 27'\ 18''.82$; and, therefore, $\angle ZV\Sigma=90^\circ-(V+\omega)=47^\circ\ 59'\ 44''.31$.

Taking into consideration *Refraction* and *Semidiameter*, we have $Z\Sigma=90^\circ\ 50'$. Ignoring the method of solution by *right-angled* spherical triangles, we obtain $\angle Z\Sigma V=28^\circ\ 30'\ 10''.876,=\angle\Sigma$. Therefore, $\frac{1}{2}(V+\Sigma)=38^\circ\ 14'\ 57''.59$, and $\frac{1}{2}(V-\Sigma)=9^\circ\ 44'\ 46''.72$; also, $\frac{1}{2}(v+\sigma)=65^\circ\ 23'\ 30''.02$. Hence $V\Sigma=120^\circ\ 13'\ 11''.21$, and $(180^\circ-V\Sigma)=59^\circ\ 46'\ 48''.79,=L$.

With the number of mean days in Hansen's sidereal year taken as the basis, we have for the Sun's mean daily motion $m=59'.136554,=59'\ 8''.19324$; and in 25 days, we have $25m=24^\circ\ 38'\ 24''.828,=M$.

The *Precessional Slip*, therefore, is $\mathbf{P}=(L-M)=35^\circ\ 8'\ 23''.96,=126503''.96$.

After adopting Struve's constant of annual precession as given in *Young's General Astronomy*, p. 528, $p=50''{\cdot}264+0''.000227(t-1900)$, and then ignoring the right-hand term, we have $\mathbf{T}=\mathbf{P}/p=2516.79$ years, or 2517 years, to be counted backward (*into the past*) from January 9, 1895, in order to determine the time of Thales.

This count brings us to January 9, 622 B. C. Dividing by the Struvian constant of *mean* annual precession, we have January 9, 628 B. C. Dividing also by the *corrected* Bessellian constant of annual precession, $50''.2479$, we have January 9, 623 B. C.

According to the *Encyclopædia Britannica*, Thales of Miletus lived from 640 B. C. to 546 B. C.; and on May 28, 585 B. C. occurred the total eclipse of the sun, which he had predicted many years before its occurrence.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

52. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

By selling a horse for $H=\$150$ cash, I gain $p=20$ per cent. At what price should I sell the horse and wait $d=90$ days, money worth $m=6$ per cent., in order to gain $q=25$ per cent.?

Solution by the PROPOSER.

Obviously the *cost* of the horse is $C=\left[1 \,/\, \left(1+\frac{p}{100}\right)\right]$ of $\$H,=\$125$;

and, consequently, the *selling price*, in order to gain q per cent., must be $P'=\left(1+\frac{q}{100}\right)$ of $\$C,=\$156.25$, which must be divided by the *proceeds* of \$1 for d days at m per cent.; that is, the *required* result becomes

$$P=\left(\frac{100+q}{100+p}\right)\left(\frac{36000}{36000-dm}\right)\text{of }\$H,=\$158.62944.$$

Also solved by *H. C. WILKES.*

53. Proposed by P. S. BERG, Larimore, North Dakota.

\$500. WOOSTER, OHIO, September 2nd, 1886.

One year after date we, or either of us, promise to pay to the order of J. M. W. Five Hundred Dollars for value received with interest at 7 per cent. from date.

J. C.
M. C.

Endorsed, May 13, 1893, \$75.00.
" September 1, 1894, \$300.00.
What was due April 1st, 1895?

Solution by H. C. WILKES, Skull Run, Virginia.

As the first payment \$75 is less than the interest then due, compute the interest to the time of the second payment.

Principal, September 2, 1886,		\$500.
Interest to September 1, 1894, 8 years less 1 day,		279.90
Interest on Interest = Interest on \$35 for 28 years less 8 days,		68.55
Amount of Principal and Interest due September 1, 1894,		\$848.45
Payments,	\$375.	
Interest on \$75 from May 13, 1893, to September 1, 1894,	7.26	
		\$382.26
Amount due September 1, 1894,		\$466.19
Interest for 7 months,		19.03
Amount due April 1st, 1895,		\$485.22

PROBLEMS.

56. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A, B, and C can walk at the rate of a 3, b 4, and c 5 miles, per hour. They start from Washington, at m 1, n 2, and p–3 o'clock, P. M., respectively. When B overtakes A, he is ordered (by A) back to C. When will B and C meet? Suppose B had ordered A back to C, when would A and C meet? In case all three continue walking ahead, at what time will they meet?

57. Proposed by L. B. FRAKER, Weston, Ohio.

Suppose that in a meadow the grass is of uniform quality and growth and that 6 oxen or 10 colts could eat up 3 acres of the pasture in $\frac{18}{25}$ of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require $2\frac{6}{7}$–weeks longer than 660 sheep to eat up 9 acres.

It what time could an ox, a colt, and a sheep together eat up an acre of the pasture on the supposition that 589 sheep eat as much in a week as 6 oxen and 11 colts? By Arithmetic, if possible.—Hunter's Arithmetic. (Unsolved in *School Visitor.*)

Solutions of these Problems should be received on or before January 1, 1895.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

48. Proposed by SETH PRATT, C. E., Assyria, Michigan.

What is the interest of \$500 for 10 years at 10 per cent. per annum, when the intervals of compounding are infinitely small?

I. Solution by Professor E. W. MORRELL, Department of Mathematics, Montpelier Seminary, Montpelier, Vermont; and BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

The formula for P dollars at compound interest for n years payable q times a year at rate r, is $P(1+\frac{r}{q})^{qn}$. In this case q is infinity; let $\frac{r}{q}=\frac{1}{x}$, whence $q=rx$ and the formula becomes $P(1+\frac{1}{x})^{xnr}$, but $(1+\frac{1}{x})^x=e$ at the limit and we have the amount$=Pe^{nr}$ and the interest will be $Pe^{nr}-P$. In this case $P=500$, $e=2.718281828$, $n=10$, $r=.10$, and we have the interest$=500(2.718281828-1)=\$859.1409142$.

II. Solution by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana; and JOHN M. ARNOLD, Crompton, Rhode Island.

$\frac{A}{P}=(1+\frac{r}{q})^{qn}=\frac{A}{500}=(1+\frac{1}{10q})^{10q}$. Expanding the second member, and reducing, $\frac{A}{500}=1+1+\frac{(1-\frac{1}{10q})}{2!}+\frac{(1-\frac{1}{10q})(1-\frac{2}{10q})}{3!}+\dots\dots\dots\dots$

When the intervals are infinitely small the number of intervals (q) is infinitely large, and the fraction in each factor of the numerator of each term is zero. $\therefore \frac{A}{500}=1+1+\frac{1}{2!}+\frac{1}{3!}+\dots\dots\dots\dots$

The sum of this series is the Naperian base. $\therefore \frac{A}{500}=2.718281828$. $\therefore A=1359.140914$, and $A-P=\$859.140914$=interest required.

III. Solution by Professor J. SCHEFFER, Hagerstown, Maryland.

Let y be the amount, a the initial principal, r the rate per cent., and t the time in years; then, we have from $dy=\frac{rydt}{100}$, $y=Ce^{\frac{rt}{100}}$, but since for $t=0$, $y=a$, we have $C=a$. $\therefore y=ae^{\frac{rt}{100}}$, and interest$=a(e^{\frac{rt}{100}}-1)$. For $r=10$, $t=10$, we have interest$=500(e-1)=500\times1.718281828=\$859.140914$.

Also solved by *O. W. ANTHONY, P. S. BERG, F. P. MATZ, C. D. SCHMITT, H. C. WILKES*, and *G. B. M. ZERR*.

49. Proposed by P. S. BERG, Larimore, North Dakota.

A man having lent \$6000 at 6 per cent. interest *payable quarterly*, wishes to receive his interest in *equal* proportions monthly, and in *advance*. How much ought he to receive monthly?

Let $\$x$=the sum he should receive monthly. But $6000\times.015=\$90=$ quarterly interest. $\therefore$ $1.015x+1.01x+1.005x=\$90$. $\therefore$ $3.03x=\$90$. $x=\$29.70297+$.

Also solved by *P. S. BERG*, *F. P. MATZ*, *J. SCHEFFER*, and *G. B. M. ZERR*.

NOTE.—Solutions of Nos. 46 and 47, Algebra, were received too late for selection from Prof. Benj. F. Yanney, A. M., Mount Union College, Alliance, Ohio.

PROBLEMS.

56. Proposed by D. G. DORRANCE, Jr., Camden, Oneida County, New York.

Sum the series 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, etc., to n terms; also what is the n^{th} term?

57. Proposed by DAVID EUGENE SMITH, Ph. D., Professor of Mathematics, Michigan State Normal School, Ypsilanti, Michigan.

Prove that the product of the n n^{th} roots of 1 is $+1$ or -1 according as n is odd or even. Prove, and generalize, for the n n^{th} roots of m.

58. Proposed by ROBERT JUDSON ALEY, M. A., Associate Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, California.

Telegraph poles are a yards apart; for how many minutes must one count poles in order that the number of poles counted may be equal to the number of miles per hour that the train is running?

Solutions of these Problems should be received on or before January 1, 1896.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

45. Proposed by GEORGE E. BROCKWAY, Boston, Massachusetts.

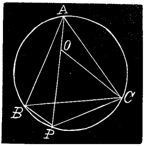
If an equilateral triangle is inscribed in a circle, the sum of the squares of the lines joining any point in the circumference to the three vertices of the triangle is constant.

Solution by JAMES F. LAWRENCE, Breckenridge, Missouri.

Let ABC be the inscribed equilateral triangle and P any point in the circumference of the circle. Join P with the points A, B, and C.

Then $AB^2=BP^2+AP^2-2BP\times AP\cos60^\circ$

$=BP^2+AP^2-BP\times AP$, and



$$AC^2 = CP^2 + AP^2 - 2CP \times AP\cos 60^\circ$$
$$= CP^2 + AP^2 - CP \times AP.$$

$\therefore\ AB^2 + AC^2 = BP^2 + AP^2 + CP^2 + AP^2 - [BP \times AP + CP \times AP]$.

But $AP = BP + PC$, American Mathematical Monthly, Vol. I., No. 9, p. 315, Prob. 19.

$AP^2 = BP \times AP + PC \times AP$, by multiplying both sides of the above equation by AP.

$\therefore\ AP^2 - [BP \times AP + PC \times AP] = 0$.

$\therefore\ AB^2 + AC^2 = BP^2 + AP^2 + CP^2$, and

$BP^2 + AP^2 + CP^2$ is constant. Q. E. D.

Excellent solutions of this Problem were received from *P. S. BERG, G. B. M. ZERR, O. W. ANTHONY, COOPER D. SCHMITT, J. F. W. SCHEFFER, JOHN B. FAUGHT, G. I. HOPKINS*, and *E. W. MORRELL*. Two solutions were received without the names of the authors signed to them.

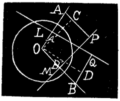
46. Proposed by J. C. GREGG, Superintendent of Schools, Brazil, Indiana.

Given two points A and B and a circle whose center is O: show that the rectangle contained by OA and the perpendicular from B on the polar of A, is equal to the rectangle contained by OB and the perpendicular from A on the polar B.

Solution by JOHN B. FAUGHT, A. B., Instructor in Mathematics, Indiana University, Bloomington, Indiana; P. S. BERG, Larimore, North Dakota; and J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Let L be the polar of A, and M the polar of B. Let AP be a perpendicular on M, and BA a perpendicular on L.

Draw OC parallel to M, and OD parallel to L. Then $OA.OA' = OB.OB' = R^2$, by definition.



$$\therefore \frac{OA}{OB} = \frac{OB'}{OA'} = \frac{CP}{BA}.$$

The triangles OAC and OBD are similar.

$$\therefore \frac{OA}{OB} = \frac{AC}{BD} = \frac{CP}{DA} = \frac{AC+CP}{BD+DA} = \frac{AP}{DA}.$$

$\therefore\ OA.BA = OB.AP$. Q. E. D.

Excellent analytical solutions of this problem were received from *G. B. M. ZERR, COOPER D. SCHMITT*, and *E. W. MORRELL*. Prof. Morrell sent in two solutions.
A solution was also received without the author's name signed to it.

PROBLEMS.

52. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

If the *center* of a rolling ellipse *move in a horizontal line*, determine the surface on which the ellipse rolls.

53. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A pole, a certain length of whose top is painted white, is standing on the side of a hill. A person at A observes that the white part of the pole subtends an angle equal to α and on walking to B, a distance a, directly down the hill towards the foot of the pole the white part subtends the same angle. What is the length of the white part, if the point B is at a distance b from the foot of the pole?

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

22. Proposed by MOSES C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Indiana.

Solve the Differential Equation,

$$(6x^3+20x^2-2x)\frac{d^2y}{dx^2}-(9x^2+10x+1)\frac{dy}{dx}+(1+9x)y=0.$$

Solution by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Dividing the given equation by the coefficient of d^2y/dx^2, then representing the coefficient of dy/dx by Q and that of y by Q', we obtain

$$\frac{d^2y}{dx^2}-Q\frac{dy}{dx}+Q'y=0\ldots\ldots(1).$$

In order to reduce (1) to the *typical* form of the text-books $F\left(\frac{dw}{dx},\ \frac{d^2w}{dx^2}\right)$,

assume $w=\frac{y}{x+1}\ldots\ldots(a)$; then $\frac{dy}{dx}=(x+1)\frac{dw}{dx}+w$, and $\frac{d^2y}{dx^2}=(x+1)\frac{d^2w}{dx^2}+2\frac{dw}{dx}$.

By substituting in (1), expanding, and reducing, we have

$$\frac{d^2w}{dx^2}=-\left(\frac{2}{x+1}-Q\right)\frac{dw}{dx}\ldots\ldots(2).$$

By putting $dw/dx=p$, (2) becomes

$$\frac{dp}{p}=-\left(\frac{2}{x+1}-Q\right)dx\ldots\ldots(3),$$

in which the variables are *separated.* Integrating each member of (3),

$$\log p_1 = \log \frac{dw}{dx} = \log\left[\frac{C(3x^2+10x-1)}{x^{\frac{1}{2}}(x+1)^2}\right] \ldots\ldots (4).$$

After passing to exponentials, etc., (4) becomes

$$w = C\int \frac{(3x^2+10x-1)dx}{x^{\frac{1}{2}}(x+1)^2} \ldots\ldots (5).$$

In order to integrate (5), put $x=\tan^2\phi$; then

$$w = 2C\left[3\int \sec^2\phi d\phi + 4\int d\phi - 8\int \frac{d\phi}{\sec^2\phi}\right] = 2C\left[3\tan\phi - \frac{4\tan\phi}{\tan^2\phi+1}\right]$$

$$= 2C\tan\phi\left[\frac{3\tan^2\phi-1}{\tan^2\phi+1}\right], \quad = 3Cx^{\frac{1}{2}}\left[\frac{3x-1}{x+1}\right]+c \ldots\ldots (6).$$

Equating the values of w as given in (a) and (6), we have

$$y = 2Cx^{\frac{1}{2}}(3x-1)+c(x+1) \ldots\ldots (7),$$

which is the complete primitive, or *general integral ;* and this assertion the following *proof* substantiates.

$$\text{From (7), } \frac{y-2Cx^{\frac{1}{2}}(3x-1)}{x+1} = c \ldots\ldots (8).$$

Differentiating (7),

$$\frac{dy}{dx} - C\left(\frac{9x-1}{x^{\frac{1}{2}}}\right) = c \ldots\ldots (9).$$

Differentiating (9),

$$2x^{\frac{3}{2}}\frac{d^2y}{dx^2} / (9x+1) = C \ldots\ldots (10).$$

Eliminating C from (9), by means of (10),

$$\frac{dy}{dx} - 2x\left(\frac{9x-1}{9x+1}\right)\frac{d^2y}{dx^2} = c \ \ldots\ (11).$$

Eliminating C from (8), by means of (10),

$$\frac{y}{x+1} - 4x^2\left(\frac{3x-1}{(9x+1)(x+1)}\right)\frac{d^2y}{dx^2} = c \ldots\ldots (12).$$

Equating the values of c as given in (11) and (12),

$$\left(\frac{4x^2(3x-1)-2x(9x-1)(x+1)}{(9x+1)(x+1)}\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{y}{x+1}=0\ldots\ldots(13).$$

$$\therefore\ 2x(3x^2+10x-1)\frac{d^2y}{dx^2}-(9x+1)(x+1)\frac{dy}{dx}+(9x+1)y=0,$$

which is the Differential Equation given by the Praposer of the problem.

Scholium.—The proposed Differential Equation is satisfied by the equations, $y=1+x\ldots\ldots(\alpha)$ and $y=2x^{\frac{1}{2}}(3x-1\ldots\ldots(\beta)$; that is, these equations are particular solutions, or *particular integrals*, of the Differential Equation.

36. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

II. Solution by the PROPOSER.

The edges adjacent to the axis of revolution generate cones; the two edges not adjacent to the axis generate a hyperboloid of one nappe. Take the axis of revolution as the axis of z. The equation of the cones is $x^2+y^2=2(z\pm\frac{1}{2}\sqrt{3}a)^2$, the altitude of each being $\frac{1}{3}\sqrt{3}a$, the radius of each being $\frac{1}{3}\sqrt{6}a$, the volume of each being $\frac{2\pi\sqrt{3}}{27}a^3$.

The equation of the hyperboloid is $2x^2+2y^2-4z^2=a^2$, the volume being the integral of $\pi(2z^2+\frac{a^2}{2})dz$ between the limits $\frac{1}{6}a\sqrt{3}$ and $-\frac{1}{6}a\sqrt{3}$, this volume being $\frac{5\pi\sqrt{3}}{27}a^3$. Adding the volume of the hyperboloid to the volume of the two cones, the total volume is found to be $\frac{\pi}{\sqrt{3}}a^3=1.8138a^3$.

[This solution is given for comparison with that of *DR. ZERR*, published in last issue. EDITOR.]

38. Proposed by L. B. FILLMAN, St. Petersburg, Pennsylvania.

The diameter of the circular base of a dome is $10\sqrt{a}$ feet, which is also the distance from any point on the circumference of the base to any point on the opposite side of the dome from base to apex. Find the volume of the dome.

I. Solution by GEORGE B. McCLELLAN ZERR, M. A., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; and the PROPOSER.

Let $x^2+y^2=a^2$ be the equation to the circle that forms a section of the dome perpendicular to the base. Then any section parallel to the base at dis-

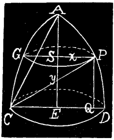
tance y from the base has a radius$=(\sqrt{a^2-y^2}-\frac{1}{2}a)$.

$$\therefore\ v=\pi\int_0^{\frac{1}{2}a\sqrt{3}}\left(\sqrt{a^2-y^2}-\tfrac{1}{2}a\right)^2dy=\frac{\pi a^3}{24}(9\sqrt{3}-4\pi),$$

$$=\frac{125\pi}{3}(9\sqrt{3}-4\pi),\text{ when } a=10,$$

$$=395.59 \text{ cubic feet.}$$

II. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland; and P. S. BERG, Larimore, North Dakota.



$CD=CP=CA=2a$; $AE=\sqrt{4a^2-a^2}=a\sqrt{3}$. Let circular section PG be at distance $ES=y$, $PS=x$; then $(a+x)^2+y^2=4a^2$. $\therefore x=-a+\sqrt{4a^2-y^2}$.

$$\text{Capacity}=\pi\int_0^{a\sqrt{3}}x^2dy=\pi\int_0^{a\sqrt{3}}(5a^2-y^2-2a\sqrt{4a^2-y^2})dy$$

$$=\frac{\pi a^3}{3}(9\sqrt{3}-4\pi),\text{ and for } a=5,\ \frac{125\pi}{3}(9\sqrt{3}-4\pi).$$

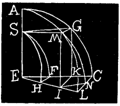
III. Solution by FRANKLIN PIERCE MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Let $AC=PC=DC=r,=10$ feet; $\angle ACD=\phi$, $=60^\circ=\frac{1}{3}\pi$; $\angle PCD=\theta$; $EQ=SP=x$; and $PQ=SE=y$. By well-known principles, then, we have $CE=r\cos\phi$; and, therefore, from the right-angled triangle CPQ, since $CQ=(r\cos\phi+x)$, we obtain the equation $(r\cos\phi+x)^2+y^2=r^2$ ...... (1). $\therefore x^2=r^2(\cos\theta-\cos\phi)^2$ ......(2). Now, $y=r\sin\theta$. $\therefore dy=r\cos\theta d\theta$......(3); also, from Calculus, $V=\pi\int x^2dy$......(4). Transforming (4) by means of (2) and (3), we obtain

$$V=\pi r^3\int_0^\phi(\cos\theta-\cos\phi)^2\cos\theta d\theta=\tfrac{1}{12}\pi r^3(\sin3\phi+9\sin\phi-12\phi\cos\phi)\ ......(5),$$

which is undoubtedly the *simplest* general result obtainable. Reducing (5) under the supposition that $\phi=60^\circ=\frac{1}{3}\pi$, we have the well-known result $V=\frac{1}{24}\pi(9\sqrt{3}-4\pi)r^3$, $=395.59+$ cubic feet.

IV. Solution by W. L. HARVEY, 75 Gray Street, Portland, Maine.



The arc of the dome$=60^\circ$ with radius$=10=a$; hence, the sine$=5\sqrt{3}=\frac{1}{2}a\sqrt{3}$, and this arc revolved about this sine generates the dome.

The quadrant CEA, revolving about EA, generates a hemi-sphere; ESF is a semi-segment of radius $=EA$. If the quadrant revolve a small distance, C

moving to L and G falling on M, and generating $CEALE$, and through MI a plane be passed parallel to CEA, the semi-segment $CGK=EFS$ generates $CKILMG$, of which the part $INLM=EFHS$.

The volume generated by the semi-segment EFS in an entire revolution will equal that generated by CGK *minus* the sum of the solids $CKNIGM$ lying about the circumference of the base of the hemi-sphere. But $GMCKIN=KI\times$ area of the semi-segment CGK; and the sum of all these parts is equal to the circumference described by EK as radius into the same area. If $EA=a$, $EI=c$, $ES=s$, and the arc $SF=p$, we obtain for the solid generated by $EAGK$, $\frac{2\pi}{3}(sc^2+a^3-a^2s)$. Consequently the solid generated by $CGK=\frac{2\pi}{3}(a^2s-sc^2)$. Then the sum of all the solids $CNIKMG$=semi-segment $GCK\times 2\pi c=\pi(cap-sc^2)$, and the volume sought is $\frac{2\pi}{3}(a^2s-sc^2)-\pi(cap-sc^2)=\pi(\frac{sc^2}{3}+\frac{2a^2s}{3}-cap)$. Putting $c^2=a^2-s^2$ this becomes, $\pi(sa^2-\frac{s^3}{3}-cap)$. In the problem $s=\frac{a}{2}\sqrt{3}$, $c=\frac{a}{2}$, $p=a(60^\circ)=\frac{a\pi}{3}$. Then $cap=\frac{a^3\pi}{2.3}$, and the contents are $\frac{a^3\pi}{2}\left(\frac{3\sqrt{3}}{4}-\frac{\pi}{3}\right)=395.59$ cubic feet. This method of solution was suggested by a solution of a similar problem by Professor Seyford, of Colby University.

[From the MONTHLY of October, 1894, pp. 257-8, we have *four* other different solutions of a similar problem; and *each* solution gives the result, $V=\frac{1}{3}\pi(9\sqrt{3}-4\pi)a^3=\frac{125}{3}\pi(9\sqrt{3}-4\pi)=395.59027+$ cubic feet. EDITOR.]

39. Proposed by J. C. GREGG, Brazil, Indiana.

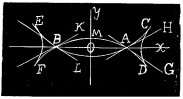
Show that the curve

$$x=9a\sin\theta-4a\sin^3\theta$$
$$y=-3a\cos\theta+4a\cos^3\theta$$

is symmetrical to the axes, and has double points and cusps: find the lengths of the arcs, (a) between the double points, (b) between a double point and a cusp, (c) and the arc connecting two cusps and not passing through a double point. [*Johnson's Calculus.*]

Solution by GEORGE B. McCLELLAN ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

The equations as given in *Johnson's Calculus* are



$x=9a\sin\theta-4a\sin^3\theta=6a\sin\theta+a\sin3\theta$,

$y=-3a\cos\theta+4a\cos^3\theta=a\cos3\theta$.

$\therefore r^2=x^2+y^2=a^2+72a^2\sin^2\theta-48a^2\sin4\theta$;

$dx=6a\cos\theta+3a\cos3\theta$, $dy=-3a\sin3\theta$.

$ds=\sqrt{dx^2+dy^2}=3a(4\cos^2\theta-1)$.

$\therefore s=3a(\theta+\sin2\theta)$.

$$\frac{dy}{dx}=-\frac{\sin 3\theta}{2\cos\theta+\cos 3\theta}=\frac{0}{0},\text{ when }\theta=\frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}.$$

$\therefore$ There are cusps at C, D, E, F. When $\theta=\frac{\pi}{6}$ and $\frac{5\pi}{6}$, $x=4a$, $y=0$, $dy/dx=\mp\frac{1}{\sqrt{3}}$; when $\theta=\frac{7\pi}{6}$ and $\frac{11\pi}{6}$, $x=-4a$, $y=0$, $dy/dx=\pm\frac{1}{\sqrt{3}}$.

$\therefore$ A and B are double points. The curve with the tangents at its double points is given in the figure. It is symmetrical to the axes, has four cusps and two double points.

$$(a),\ s=AMB=3a\Big[\theta+\sin 2\theta\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}=3a\left(\frac{\pi}{3}+\sqrt{3}\right)=a(\pi-3\sqrt{3}),$$

$$(b),\ s=AD=3a\Big[\theta+\sin 2\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\tfrac{1}{2}\pi a,$$

$$(c),\ s=DC=3a\Big[\theta+\sin 2\theta\Big]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}=a(3\sqrt{3}-\pi).$$

Whole length of curve $=2AMB+4AD+2CD=2a(\pi+6\sqrt{3})$.

[The given curve is one of the involutes of a 4-cusped hypocycloid, which could be drawn surrounding the figure. The curve as given by the equations as first proposed is symmetrical to the axes, has two double points but no cusps. EDITOR.]

PROBLEMS.

47. Proposed by Professor J. SCHEFFER, A. M., Hagerstown, Maryland.

The floor of a vault forms a square, and all sections parallel to it are squares. The two vertical sections through the middle points of the opposite sides of the floor are equal semi-circles. Find the convex surface and the volume of the vault.

48. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

I have a circular section basin 12 inches in perpendicular height; the diameters are as follows: At base, 2 inches; one inch perpendicular height, 6 inches; two inches perpendicular height, 18 inches; three inches perpendicular height, 54 inches; and so on, the diameter being trebled for every inch in height. After a rain the water in the basin is six inches deep, what was the rainfall?

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

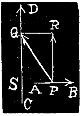
SOLUTIONS OF PROBLEMS.

28. Proposed by O. W. ANTHONY, Professor of Mathematics, New Windsor College, New Windsor, Maryland.

A movable finite wire carrying a current of electricity is perpendicular to and on one side of an infinite wire also carrying a current. Investigate the motion of the movable wire.

Solution by the PROPOSER.

Let AB be the finite wire, and DC the infinite wire. Let the current flow away from the infinite wire in the short one. Also call μ_1, μ_2 the current strengths of the two currents, and m the power of the finite wire. Then $df_{PQ}=\mu_1\mu_2\frac{dxdz}{x^2+z^2}$, $x=QS$, $z=PS$. Resolving forces perpendicular and parallel to DC we have



$$df_{PR}=\mu_1\mu_2\frac{xdxdz}{(x^2+z^2)^{\frac{3}{2}}}.$$

$$\therefore f_{PR}=2\mu_1\mu_2\int_{z_1}^{z_2}\int_0^{\infty}\frac{xdxdz}{(x^2+z^2)^{\frac{3}{2}}}$$

$$=2\mu_1\mu_2\log\left(\frac{z_2}{z_1}\right).$$

$$\therefore \frac{d^2s}{dt^2}=\frac{2}{m}\mu_1\mu_2\log\left(\frac{z_2}{z_1}\right).$$

$$s=\frac{1}{m}\mu_1\mu_2\log\left(\frac{z_2}{z_1}\right)t^2+k_1t+k_2.$$

29. Proposed by J. A. CALDERHEAD, A. B., Superintendent of Schools, Limaville, Ohio.

Show that if a body be projected from the angle A of a plane triangle ABC so as to strike the side CB at a point D, then, if its course after reflection at D be parallel to AB, $\tan\angle DAB=\frac{(1+e)\cot B}{1-e\cot^2 B}$.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi, University P. O., Mississippi.

The angle between the course of the body before impact and the side CB is $180^\circ-(B+DAB)$.

$$\therefore e=\frac{\tan B}{\tan[180^\circ-(B+DAB)]};$$

$$-\tan B = e\,\frac{\tan B + \tan A}{1 - \tan B \tan A};$$

$$\tan DAB = \frac{(1+e)\tan B}{\tan^2 B - e}$$

$$= \frac{(1+e)\cot B}{1 - e\cot^2 B}.$$

Also solved by *O. W. ANTHONY*, and *J. SCHEFFER*.

PROBLEMS.

35. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

A man weighs 150 pounds; his balloon with all its attachments weighs 500 pounds. What volume of pure hydrogen must be made and put into the balloon so that it will be on the point of ascending with the man? How many kilograms of zinc and of hydrogen sulphate will be used in generating the hydrogen? Give volume of hydrogen in cubic feet given that one litre of hydrogen weighs .0896 grams.

36. Proposed by O. W. ANTHONY, Professor of Mathematics, New Windsor College, New Windsor, Maryland.

A vertical slit is made in the middle of the side of a rectangular box containing water. What is the time required to empty the box?

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

26. Proposed by J. W. WATSON, Middlecreek, Ohio.

Find the average area of all right-angled triangles having a *constant* hypotenuse.

III. Solution by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

In my first two solutions I made an arm of the right-angled triangle vary uniformly, although I employed two different systems of co-ordinates. I continued this variation until the arms of the right-angled triangle became equal; and by doing this I avoided all *reciprocal* equal right-angled triangles. Looking at results from this standpoint, the verdict must be—*correct*. By simply varying

uniformly an arm of an inscribed right-angled triangle, as I have done, I am now fully convinced that not all the *possible* right-angled triangles are comprehended; for, most certainly, the uniform variation of an arm does not cause a uniform variation of the vertex of the right angle—along a quadrant of the circumscribing circle. I say the *most natural* solution of this problem is the one in which the *arc* of the circumscribing circle is made to vary uniformly, and this variation is to extend over only one quadrant of the circumscribing circle. Let x=the arc intercepted by an arm and the constant hypotenuse h; then, if $\frac{1}{4}\pi h=a$, the required average area is

$$A=\tfrac{1}{2}h^2\int_0^a \sin\left(\frac{x}{h}\right)\cos\left(\frac{x}{h}\right)dx\div\int_0^a dx=\frac{h^2}{2\pi}\cdots\cdots(1).$$

Fourth Solution.

Taking θ as the *central* angle intercepted by an arm, and by the constant hypotenuse, of the right-angled triangle, the required average area becomes

$$A=\tfrac{1}{4}h^2\int_0^{\frac{1}{2}\pi}\sin\theta d\theta\div\int_0^{\frac{1}{2}\pi}d\theta=\frac{h^2}{2\pi}\cdots\cdots(2).$$

Fifth Solution.

Making ϕ one of the acute angles, the required average area becomes

$$A=\tfrac{1}{4}h^2\int_0^{\frac{1}{4}\pi}\sin 2\phi d\phi\div\int_0^{\frac{1}{4}\pi}d\phi=\frac{h^2}{2\pi}\cdots\cdots(3).$$

Sixth Solution.

Let the origin of Cartesian co-ordinates be placed at the center of the circle; then the required average area becomes

$$A=\tfrac{1}{4}h^2\int_0^{\frac{1}{2}h}dx\div\tfrac{1}{2}h\int_0^{\frac{1}{4}\pi}\frac{dx}{\sqrt{(\frac{1}{4}h^2-x^2)}}=\frac{h^2}{2\pi}\cdots\cdots(4).$$

Several other solutions leading to the same result are possible.

NOTE.—Professor O. W. Anthony sent us a note in which he defends the solution leading to $\frac{1}{4}a^2$ as the answer. His argument being, in substance, this: The mind does not form a picture of a right triangle inscribed in a semi-circle whose diameter is a but simply a right triangle whose hypotenuse is a. He therefore concludes that the number of triangles should be found by varying one of the sides.

The discussion of this problem called forth the excellent article, "A Note on Mean Values," by Dr. Moore, page 303 of this issue of the MONTHLY. I am quite sure that that article will be greatly appreciated by those of our readers who are interested in this abstruce subject, Mean Value.

Taking the substance of that article as criterion, it remains to determine whether or not the above problem is stated in the *definite* form. I hold the opinion that it is stated in the *definite* form; for the problem requires the average area of *all* right triangles having a given hypotenuse. It does not require the average area of all right triangles having a given hypotenuse and formed according to a certain law, but all the right triangles having a given hypotenuse and the law of formation must be so chosen as to give *all* such right triangles. Therefore, the solutions leading to the result $\frac{a^2}{2\pi}$ are the *correct* and *only* solutions of the problem. EDITOR.]

PROBLEMS.

35. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the chance that the distance of two points within a square shall not exceed a side of the square. [*Byerly's Integral Calculus, p. 209.*]

36. Proposed by O. W. ANTHONY, Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

A box contains n^2 blocks numbered from 1 to n^2. What is the probability that two consecutive numbers will be adjacent ?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

DIOPHANTUS' EPITAPH.

Translated by Rev. WRIGHT G. CAMPBELL, A. M., Professor of Ancient Languages, Irving College, Mechanicsburg, Pennsylvania.

Here Diophantus has a mound which to you, with wonderful art, signifies the times of his life.

One-sixth part he lived a youth ; then, in the twelfth part, slowly he began to clothe his cheeks.

In the seventh part after these, he was joined to a wife ; and in the fifth year, a beautiful boy was born.

After he had attained half of the paternal age, he unhappy seized by sudden death, died.

Four summers the surviving father was compelled to mourn.

From this you may ascertain his years. *Contributed by F. P. Matz.*

SOLUTIONS OF PROBLEMS.

NOTE.—The comment on page 285 of last issue should have been credited to M. A. Gruber, War Department, Washington, D. C.

27. Proposed by J. W. NICHOLSON, LL. D., President, and Professor of Mathematics, Louisiana State University and A. and M. College, Baton Rouge, Louisiana.

Required a formula for finding five integers the sum of whose cubes is a cube.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

A very simple formula is obtained as follows:

$$x^3+(6x)^3+(8x)^3=(9x)^3\ldots\ldots(1);\quad (3x)^3+(4x)^3+(5x)^3=(6x)^3\ldots\ldots(2).$$

(2) in (1) gives $x^3+(3x)^3+(4x)^3+(5x)^3+(8x)^3=(9x)^3$, where x can have any value, positive, integral.

II. Solution by H. W. DRAUGHON, Olio, Mississippi.

If, in the identity, $(x+y)^3=x^3+3x^2y+3xy^2+y^3\ldots\ldots(1)$, we make $3x^2y$ a cube, and $3xy^2$, the sum of two cubes, we will have a formula for finding five numbers the sum of whose cubes is a cube.

Put $3x^2y=m^3x^3$; whence $x=\dfrac{3y}{m^3}$, and $3xy^2=\dfrac{9y^3}{m^3}=\left(\dfrac{2y}{m}\right)^3+\dfrac{y^3}{m^3}$.

By substitution, (1), becomes,

$$\left(\frac{3y}{m^3}+y\right)^3=\left(\frac{3y}{m^3}\right)^3+\left(\frac{3y}{m^2}\right)^3+\left(\frac{2y}{m}\right)^3+\left(\frac{y}{m}\right)^3+y^3.$$

Multiplying out by m^9, we have

$$(3+m^3)^3y^3=27y^3+27m^3y^3+8m^6y^3+m^6y^3+m^9y^3\ldots\ldots(A),$$

which is the formula required. Let $y=1$, and $m=4$; then we have,

$$(3)^3+(12)^3+(32)^3+(16)^3+(64)^3=(67)^3.$$

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

Let x_1, x_2, x_3, x_4, x_5, be the integers. Then

$$x_1^3+x_2^3+x_3^3+x_4^3+x_5^3=x^3\ldots\ldots(1).$$

Let $x_1=3m$, $x_2=4m$, $x_3=5m$. Then (1) may be written

$$(6m)^3+x_4^3+x_5^3=x^3\ldots\ldots(2),\text{ or } z^3+x_4^3+x_5^3=x^3\ldots\ldots(3).$$

Let $z=3u$, $4u$, or $5u$, $x_4=4u$, $5u$, or $3u$, $x_5=5u$, $3u$, or $4u$.

We have $6m=3u$; $6m=4u$; $6m=5u$. Then $u=2m$; $u=\dfrac{3m}{2}$; $u=\dfrac{6m}{5}$.

Therefore,

$$\left.\begin{array}{l} x_1=3m, \\ x_2=4m, \\ x_3=5m, \\ x_4=8m, \\ x_5=10m, \end{array}\right| \text{I.} \qquad \text{Or} \qquad \left.\begin{array}{l} x_1=3m, \\ x_2=4m, \\ x_3=5m, \\ x_4=\frac{15m}{2}, \\ x_5=\frac{9m}{2}, \end{array}\right| \text{II.}$$

$$\text{Or} \qquad \left.\begin{array}{l} x_1=3m, \\ x_2=4m, \\ x_3=5m, \\ x_4=\frac{18m}{5}, \\ x_5=\frac{24m}{5}, \end{array}\right| \text{III.}$$

Changing II, III, to a form which shall always be integral, we have the following table of formulæ :

$x_1=3m,\ 6m,\ 15m.$
$x_2=4m,\ 8m,\ 20m.$
$x_3=5m,\ 10m,\ 25m.$
$x_4=8m,\ 15m,\ 18m.$
$x_5=10m,\ 9m,\ 24m.$ Where m has any integral value.

28. Proposed by DAVID EUGENE SMITH, Ph. D., Professor of Mathematics, Michigan State Normal School, Ypsilanti, Michigan.

Decompose the product 97.673.257 into the sum of two fourth powers.

I. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Solution by Determinants :

$$97=\begin{vmatrix} 9 & -4 \\ 4 & 9 \end{vmatrix},\quad 673=\begin{vmatrix} 23 & -12 \\ 12 & 23 \end{vmatrix},\quad 257=\begin{vmatrix} 16 & -1 \\ 1 & 16 \end{vmatrix}.$$

$$\text{By multiplication } 673.97=\begin{vmatrix} 23 & -12 \\ 12 & 23 \end{vmatrix}\times\begin{vmatrix} 9 & -4 \\ 4 & 9 \end{vmatrix}=\begin{vmatrix} 255 & -16 \\ 16 & 255 \end{vmatrix}.$$

$$\text{and } \begin{vmatrix} 255 & -16 \\ 16 & 255 \end{vmatrix}\times\begin{vmatrix} 16 & -1 \\ 1 & 16 \end{vmatrix}=\begin{vmatrix} 4080-16, & 255-256 \\ 256-255, & 16+4080 \end{vmatrix}$$

$$=\begin{vmatrix} 4096 & -1 \\ 1 & 4096 \end{vmatrix}=\begin{vmatrix} 64^2 & -1 \\ 1 & 64^2 \end{vmatrix}=64^4+1^4.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

$$97.673.257=(3^4+2^4)(23^2+12^2)(4^4+1^4).$$

$$(a^4+b^4)(c^4+d^4)=(a^2c^2\pm b^2d^2)^2+a^2d^2\mp b^2c^2)^2=A^2+B^2.$$

$$(a^4+b^4)(c^4+d^4)(e^2+f^2)=(A^2+B^2)(e^2+f^2)=(Ae\pm Bf)^2+(Af\mp Be)^2$$

$$=\{(a^2c^2\pm b^2d^2)e\pm(a^2d^2\mp b^2c^2)f\}^2+(a^2c^2\pm b^2d^2)f\mp(a^2d^2\mp b^2c^2)e\}^2.$$

Let $a=3$, $b=2$, $c=4$, $d=1$, $e=23$, $f=12$.

$\therefore$ $97.673.257=1^2+4096^2=3359^2+2344^2=3041^2+2744^2=1511^2+4064^2$.

Of these four sums, the first is the only one that fulfills the condition.

$\therefore$ $97.673.257=1^4+64^4$

NOTE.—*A. H. BELL, H. W. DRAUGHON, J. C. CORBIN,* and *F. P. MATZ* should have been credited with solutions of No. 22.

PROBLEMS.

40. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The sum of the three positive integral *cubic* roots of an equation is a square. What is the equation ?

41. Proposed by H. C. WILKES, Skull Run, West Virginia.

Given $\dfrac{50(a+b)}{ab}=\dfrac{81(c+d)}{cd}$......(1) ; $\dfrac{56(a+c)}{ac}=\dfrac{75(b+d)}{bd}$......(2) ; $\dfrac{65(b+c)}{bc}=\dfrac{66(a+d)}{ad}$......(3), to find the least integral values of a, b, c, d.

42. Proposed by E. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

In a parallelogram, sides a and b, diagonals c and d, $2a^2+2b^2=c^2+d^2$. Find all the parallelograms, not rectangles, whose sides and diagonals are rational.

Examples :

a	b	c	d
4	7	9	7
16	7	21	13
8	9	13	11
8	11	17	9

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

28. Proposed by "IAGO"—(The late DR. JAMES MATTESON, DeKalb Center, Illinois.)

If 9 gentlemen, or 15 ladies, will eat 17 apples in 5 hours, and 15 gentlemen and 15 ladies can eat 47 apples of a similar size in 12 hours, the apples growing uniformly; how many boys will eat up 360 apples in 60 hours, admitting that 120 boys can eat the same number as 18 gentlemen and 26 ladies? *F. P. Matz.*

Solution by Professor P. S. BERG, Larimore, North Dakota.

1. Call the original size of an apple an "apple unit."
2. Call the growth of 1 apple in 1 hour one "unit of growth."
3. $32\frac{1}{7}$ boys will in the same time eat as many as 9 gentlemen or 15 ladies.
4. $\therefore$ $32\frac{1}{7}$ boys in 5 hours or $160\frac{5}{7}$ boys in 1 hour=17 "apple units"+85 "units of growth."
5. $\therefore$ $85\frac{5}{7}$ boys in 12 hours or $1028\frac{4}{7}$ boys in 1 hour=47 "apple units"+ 564 "units of growth."
6. $\therefore$ $(4)\times 2\frac{13}{17}=444\frac{39}{119}$ boys in 1 hour=47 "apple units"+235 "units of growth."
7. $\therefore$ $(4)\times 6\frac{54}{85}=1066\frac{46}{119}$ boys in 1 hour=$112\frac{4}{5}$ "apple units"+564 "units of growth."
8. $\therefore$ $(5)-(6)=584\frac{29}{119}$ boys in 1 hour=329 "units of growth."
9. $\therefore$ $(7)-(5)=37\frac{97}{119}$ boys in 1 hour=$65\frac{4}{5}$ "apple units."
10. $\therefore$ $(9)\times 15\frac{9}{20}=584\frac{29}{119}$ boys in 1 hour=$1016\frac{61}{100}$ "apple units."
11. $\therefore$ 329 "units growth"=$1016\frac{61}{100}$ "apple units,
    1 "unit growth"=$3\frac{9}{100}$ "apple units."
12. 360 apples in 60 hours=360 "apple units"+21600 "units growth."
13. 360 "apple units"=$116\frac{56}{309}$ "units growth."
14. $\therefore$ 360 apples in 60 hours=$116\frac{56}{309}+21600=21716\frac{56}{309}$ "units growth."
15. From (8) 1 boy in 1 hour=$\frac{119\times 329}{69525}$ "units growth."
16. $\therefore$ 1 boy in 60 hours=$\frac{1566604}{4635}$ "units growth."
17. $\therefore$ $21716\frac{56}{309}\div\frac{1566604}{4635}=643$ boys.

29. Proposed by ALEXANDER MACFARLANE, M. A., D. Sc., LL. D., Professor of Electrical Engineering in Lehigh University, South Bethlehem, Pennsylvania.

A rectangular room has the four walls, the floor, and the ceiling covered with mirrors; a candle is placed inside the room: find a formula which will express all the images.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Regard the candle as a luminous point. Then, since there are three sets of parallel mirrors, we have, from elementary optics, three sets of an infinite

number of images situated on three straight lines at right angles to one another, and intersecting at the bright point.

These mirrors are also inclined to one another at right angles. Let $A^\circ=\frac{\pi}{2}=$the angle of inclination of the mirrors, a°, b° the angles made by the candle with two of the mirrors.

Then $\frac{360^\circ-(a^\circ+b^\circ)}{A^\circ}=\frac{360^\circ-90^\circ}{90^\circ}=3=$the number of images due to two of the mirrors inclined at 90°. There are twelve such sets of inclined mirrors, but of the 36 images formed, 18 are repeated. $\therefore$ ½ of 12 of $3=18$ images due to the inclined mirrors.

$\therefore\ \frac{12\{2\pi-(a^\circ+b^\circ)\}}{\pi}$, is the formula for the images due to the inclined mirrors, where $a^\circ+b^\circ=\frac{\pi}{2}$.

30. Proposed by R. J. ADCOCK, Larchland, Warren County, Illinois.

When the sum of the distances of a point of a plane surface, from all the other points, is a minimum, that point is the center of gravity of the plane surface.

I. Proof by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let (x_1, y_1) be the point, (x, y) any other point, S the sum of the distances of (x, y) from (x_1, y_1).

$$\text{Then } S=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\sqrt{(x-x_1)^2+(y-y_1)^2}\,dxdy.$$

Let $\int_{x_1}^{x_2}\int_{y_1}^{y_2}$ be represented by $\int$, and $dxdy$ by dA.

$$\therefore\ S=\int\sqrt{(x-x_1)^2+(y-y_1)^2}\,dA=\int D\,dA.$$

$$\text{For a minimum, } \frac{dS}{dx_1}=\frac{(x-x_1)dA}{D}=0,\ \frac{dS}{dy_1}=\frac{(y-y_1)dA}{D}=0.$$

$$\therefore\ (x-x_1)dA=0,\ (y-y_1)dA=0.\quad \therefore\ x_1=\frac{\int xdA}{\int dA},\ y_1=\frac{\int ydA}{\int dA}.$$

$$\therefore\ x_1=\frac{\int\int xdxdy}{\int\int dxdy},\ y_1=\frac{\int\int ydxdy}{\int\int dxdy}.$$

II. Remark by S. H. WRIGHT, M. D., M. A., Ph. D., Penn Yan, New York.

Mr. Adcock's problem asserts the truth evidently, when regular plane surfaces are considered, such as the square, rectangle, parallelogram, rhombus, the circle, and *equilateral polygons*. I hardly believe the problem will apply to *any irregular* figure.

III. Comment, etc., by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

It is evidently meant that when the sum of the squares of the distances of a point from all other points is a minimum the point is the *c. g.* of the surface. It can easily be proved that the other is not true.

[If Prof. Anthony will furnish a proof that the proposition does not hold for *any or all figures* we will be glad to publish it. We append Prof. Anthony's proof of the well-known proposition which he quotes. EDITOR.]

[The sum of the squares of the distances of a point (h, k) from all other points in the surface is $u=\int\int\left[(x-h)^2+(y-k)^2\right]dxdy$, where the integration is over the entire surface. For minimum, $\dfrac{du}{dh}=0$, $\dfrac{du}{dk}=0$. i. e.,

$$\int\int(x-h)dxdy=0, \text{ and} \int\int(y-k)dxdy=0\,;$$

$$\text{Whence } h=\frac{\int\int xdxdy}{\int\int dxdy}, \text{ and } k=\frac{\int\int ydxdy}{\int\int dxdy}.$$

That is (h, k) is the center of gravity of the surface.]

NOTE. In Prof. Ross' problem in September-October No., p. 291, read "*square* field ABC" instead of "*rectangular* field;" also insert "irregular" before the second "plane curve" in line 2 of Prof. Taylor's problem, and read "distance" for "distances" and $(C)\ h$ for $(C\ h)$ in line 5 of same problem.

PROBLEMS.

35. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude: taking the Right Ascension and Declination of the former to be 22 hours, 52 minutes, —30 degrees, 12 minutes; of the latter 16 hours, 23 minutes, —26 degrees, 12 minutes?

36. Proposed by J. K. ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pennsylvania.

"What is the length of a chord cutting off one-fifth of the area of a circle whose diameter is 10 feet?"

37. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A gentleman owned and lived in the center, R, of a rectangular tract of land whose diagonal, D, 350 rods, dividing the tract into two equal right-angled triangles, in each of which is inscribed the largest square field, F and F', possible; the north and south boundary lines of the two square fields being extended and joined formed a little rectangular lot,

R, in the center around the residence. The difference in the area of the *entire rectangular* tract and the *sum* of the areas of the two square fields, F, F, is $187\frac{1}{2}$ acres. Give the dimensions and area of the entire tract, and one square field, F.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

A POSTULATE OF THE HYPOTHESIS OF THE FOURTH DIMENSION.

Let it be granted that a straight line may be drawn through any point of the space in which our universe is contained, every other point of the supposed straight line being outside of our space.

This is *a postulate* logically involved in Arthur Willink's speculation respecting a fourth direction. He says that the fourth direction is *unknown*. He reasons that this could not be if two points in the fourth direction were posited in our space, since two given points in a straight line determine its position, and its direction becomes known.

According to Arthur Willink the direction of the fourth dimension is *unknown*. Hence, the fourth dimension can intersect our space in but *one* point. Hence, every other point on the hypothetical fourth dimensional straight line except that of the intersection must lie outside of our trinally extended space. Practically to the denizens of our universe that means that a straight line may be drawn where our space is not. This hypothesis of a fourth dimension, therefore, places restrictions upon the extent of our space, whereas no ultimate boundary is assigned to it by either the intellect or the imagination of man. The human mind reports as the result of its cognition *one illimitable space*. The hypothesis of another and a wider space is inconsistent with this cognition.

Let us view this subject in another light. Three straight lines mutually perpendicular to each other may be drawn through any point in our space, and hence through the point in which the fourth dimension is supposed to intersect it. The third dimension is perpendicular to the plane of the first and second dimensions. If this plane is definitely located, the direction of the third dimension is determined. Is the fourth dimension, also, at right angles to this plane? If so it must coincide with the third dimension and therefore lie in our space. But this conclusion contradicts the hypothesis that the fourth dimension is not in our space.

Finally, if the alleged fourth dimension is not perpendicular to the plane of the first and second dimensions is it a *legitimate* dimension?

JOHN N. LYLE.

The derivitive of $e^{-xi}(\cos x+i\ \sin x)$ is zero.

$\therefore\ e^{-xi}(\cos x+i\ \sin x)=A$, a constant.

Putting $x=0$, $A=1$.

$\therefore\ e^{xi}=\cos x+i\ \sin x$.

WM. E. HEAL, Marion, Indiana.

In $e^{xi}=\cos x+i\ \sin x$, let $x=m\pi$. $\therefore\ e^{m\pi i}=\cos m\pi+i\ \sin m\pi$. $\therefore\ m\pi i=\log [\cos m\pi+i\ \sin m\pi]$. If in this m is any positive or negative integer. the last term, $i\ \sin m\pi$, will disappear; and we may write: $m\pi i=\log\ \cos m\pi$. Now for any *even* number form, $\cos m\pi$ becomes $+1$, for any *odd* -1. $\therefore$ log $\cos m\pi=\log(\pm 1)$. $\therefore\ m\pi i=\log(\pm1)$. For $m=1$, $\pi i=\log(-1)$. We may then write: $m\log(-1)=\log(\mp 1)$ [Cf. *Schurig's Algebra*, 73, 2.z.]. The upper sign is to be taken for m even, the lower for m odd.

If $u=e^x$, $\log(\pm u)=\log u+\log(\pm1)=\log e^x+\log(\pm1)=x+m\log(-1)$. The sign to be taken as before.

OSCAR SCHMIEDEL, A. M., Professor of Mathematics,
Bethany College, Bethany, West Virginia.

WANTED.—A solution by Quaternions, giving the elementary steps in simple English, of some problem published in the MONTHLY.

A READER.

Olney, in his *General Geometry*, says "The problem of the duplication of the cube and the trisection of an angle has been shown to be identical, as both depend upon the insertion of two means in a continued proportion between two extremes." Where can I find a proof of the statement in reference to the trisection of an angle?

N. F. DAVIS, 21 George Street,
Providence, Rhode Island.

(A) The trisection of an angle: *The trisection of a right line taken as the chord of any arc of a circle trisects the angle of the arc;* (B) Duplication of the Cube: *Doubling the dimensions of a cube octuples its contents, and doubling its contents increases its dimensions twenty-five plus one per cent.*

By request of the author,
EDW. J. GOODWIN, Solitude, Indiana.

DEFINITION OF A FRACTION.

I. In compliance with your request in September No. (1894) of MONTHLY, I state that my preference of definitions of fractions is (B).

H. W. DRAUGHON.

II. I think that a fraction is an indicated division. Division has two meanings: 1, The division into parts; 2, The division by a number of the same

kind. A group of 12 cows÷4=a group of 3 cows. 12 cows÷4 cows=3 times as many.

H. C. WHITAKER.

III. The definition (A) suits me the best if only applied to "abstract" arithmetic—provided, the meaning of "a unit" is defined in advance. But as the sentence reads: "a unit or anything else," it leaves a suspicion in my mind that all mathematicians have not a fixed definition for a unit in the scientific sense. To get at what a mathematical fraction is, we must first all agree on a common basis for numeric systems. I submit my views. First, notation of any kind, or the symbols (figures) we call numbers, have *no value other* than that given by the "something" the numbers represent; second, every numerical system is founded on three principles, viz: singularity, plurality, and totality. The first principle is represented by the so-called *indefinite* one. All indefinite ones represent, in the scientific sense, indivisible fractions, of utility only in counting "items." From these indivisible ones *units* are raised, that is, the least "plural" (the second principle) is obtained by composing two of the indivisible ones into one united whole. This, the first and lowest unit, is called two. Again, by uniting the lowest unit and the indivisible fraction into a greater whole, called three, the least numeric total (the third principle) is expressed.

According to this conception of numbers, a multitude of fractions are convertible into units of divers magnitude and from a number of units, a multitude of fractions or finite ones can be evolved, but every unit used in computation *must have a fixed, definite, intrinsic* value (its magnitude) as well as a relative value depending on its combination with other units and fractions.

With this understanding of what constitutes the foundation for all scientific numerical systems (not applicable to empiric systems), the definition of a fraction may be reduced to this:—A fraction is a given part of a given unit the magnitude of which is called the denominator, and is written under a line, while the given part or parts of the unit's magnitude is written above the line and is called the numerator; hence, the numerator indicates the fraction of a certain unit designated by the denominator.

CHAS. DE MEDICI,
60 West 22d Street, New York City.

[We give all our subscribers opportunity to express *their* views *briefly* upon the questions raised in the MONTHLY whether these views are founded upon what we consider right conceptions or not. EDITORS.]

NOTE ON HELMHOLTZ'S USE OF THE TERMS "SURFACE" AND "SPACE" AS IDENTICAL IN MEANING: —Does the "immortal" Helmholtz in his Lectures on the—"Origin and Significance of Geometrical Axioms"—use the terms "surface" and "space" as identical in meaning?

If so, is not his performance "plainly pseudological" however "sickening" the statement of the fact may be to the devotees who would have us bow down before their pseudo-spherical idol and reverence their pseudological prophet as an infallible authority.

JOHN N. LYLE.

COMMENTS ON DUREGE'S THEORY OF FUNCTIONS.

Durege's book is admirably adapted for introducing beginners to the theory of functions according to the methods of Riemann. Ever since its first appearance, I have been in the habit of recommending it to my students.

PROFESSOR DR. L. FUCHS,
University of Berlin.

September 5th, 1895.

His (Durege's) success lies in his art of execution; the material chosen with rare skill is worked up into a continuous whole and treated with masterly lucidity, and the interest is constantly kept up by happy examples. The present work, the fourth edition, which was prepared by Durege shortly before his death, is excellently adapted for an introduction to the study of complex variables. Beginning with the historical development of the idea of complex quantities, the author first establishes the general properties of functions of such quantities. * * * Durege's book will certainly maintain for a long time to come the prominent position in literature which it occupies at present.

Zeitschript fur das Realschulwesen, 1894.

An instructive book, in which is developed according to the ideas of Cauchy, and especially of Riemann, the classical theory of a complex or imaginary variable. The examples are simple and well chosen. * * * A good manual for students.

Revue Generale des Sciences Pures et Appilquees, 1894.

A new edition of Dr. Durege's most excellent book, well known to every mathematician, lies before us. Compared with the third edition, only a few changes appear, consisting in short additions, more numerous examples, and alterations in style. * * * The excellencies of Durege's presentation of the elements of the theory of functions are so generally known, that they relieve the reviewer in a most agreeable way of making any commendation or criticism. We have no book which is so well adapted for the introduction to the more recent theory of functions, and for facilitating the study of the works of that great mathematician (Riemann), as the one here noticed.

Naturwissenschaftliche Wochenschript, 1894.

The fourth edition of the present work has now appeared, which fact is a conclusive proof of its excellence. As this edition, compared with the third, contains only a few changes, consisting in short additions, more numerous examples, alterations in style, etc., we may dispense with a more minute discussion of it. The work is based upon the modern theories of functions following Riemann's method, and is eminently adapted to introduce the student to the conceptions of this great mathematician—whom it is quite difficult to understand in the original. The present work has full raison d' etre of Neumanns book "Vorlesungen

uber Riemann's Theorie der Abelschen Integrale," and it is even to be recommended to a beginner in preference to the latter on account of its greater precision.

Central Organ fur die Interessen des Realschulwesens, 1894.

EDITORIALS.

No. 10, Vol. II, of the *Mathematical Magazine* is expected to be ready in January, 1896.

Prof. William Hoover should have been credited with solving Problem 17, Average and Probability. His solution was accidentally mislaid.

D. G. Durrance sent in a solution of Problem 49, Department of Arithmetic, after the July-August number had gone to press. His solution is by Algebra.

Drs. W. W. Beman and D. E. Smith have written a *Plane and Solid Geometry* which is being published by Ginn & Co. Something new and original may be expected.

Dr. G. A. Miller requests us to state that his position in Michigan University was an instructorship instead of a professorship. On account of his travels during the summer he failed to see the error until recently.

Prof. J. R. Baldwin has resigned his position as Professor of Mathematics in the Davenport Business College to accept a position at an increased salary in the Commercial Department in the High School of Davenport, Iowa.

Dr. Zerr notified us some time ago that his solution of Problem 43, Department of Geometry, is wrong. It does not follow from (3) and (4) that $y=z^2$. Professor Scheffer has pointed out the same error. We shall be pleased to publish a correct solution of this problem in the next issue of the MONTHLY.

We take pleasure in announcing that Drs. G. E. Fisher and I. J. Schwatt, of the University of Pennsylvania, have in press a translation of Durege's Theory of Functions of a Complex Variable, with special reference to Riemann surfaces. Durege's book is considered the standard text-book on the Theory of Functions. See comments on this book under Queries and Information.

The latest venture in the field of mathematical journalism in this country is the *American Mathematical Monthly*, edited by Profs. B. F. Finkel and J. M. Colaw, and issued monthly at $2. a year. The second volume is in progress and the double number for July and August, 1895, is at hand. The VISITOR heartily wishes the plucky editors the success they so richly deserve.

The Mathematical Visitor, 1895.

Prof. O. W. Anthony writes: "Allow me to congratulate you on the great success you are making of the MONTHLY. It is the best mathematical paper published for working mathematicians. I will send in my subscription for the coming year in a short time, and if there is any falling behind in financial matters, will be more than willing to bear my share." We are very thankful for Professor Anthony's kind words and generous offer of substantial support. We wish that the many hundreds of mathematicians of this country who are not now subscribers, would manifest the same spirit; they would then put their names upon our subscription list and contribute to the pages of the MONTHLY.

BOOKS AND PERIODICALS.

Elements of Geometry. By George C. Edwards, Ph. B., Associate Professor of Mathematics in the University of California. 8vo. cloth, 293 pp. Price, $1.10. New York: Macmillan & Co.

Some of the salient features of this new work are the concise and accurate statement of the definitions, the natural arrangement of the parts, the great generality of the demonstrations of many of the propositions, numerous interesting and valuable notes, and the development of method of attack in the solution of problems.

It is to be regretted that the author has omitted the subject of *Proportion*, giving as his reason that Proportion properly belongs to Algebra. While this is true, it is also true that many students begin the study of Geometry before they come to the study of Proportion in Algebra. But even if they have been drilled in the subject in Algebra, it has been my experience that the little time required for its discussion in Geometry is most helpful to even the brightest students, while its omission would prove very unsatisfactory to those who have not had it previously or who have had it several years previous to taking up Geometry.

The last chapter is devoted to the treatment of the Conic Sections. At the end of Plane Geometry and at the end of Solid Geometry there is given a large number of exercises designed to review the work preceding them, and thoroughly to establish method of attack in the mind of the student. Corollaries and scholia have been in large measure replaced by well chosen exercises. On pages 155—162 are thirty-nine diagrams to illustrate as many different demonstrations of the Pythagorean Proposition. The book is well written and the publishers have presented it for public favor in good style. B. F. F.

Plane and Spherical Trigonometry. By G. A. Wentworth, A. M., author of a series of text-books in Mathematics. Revised edition. 8vo. cloth and leather back, 192 pp. Price, $0.85. Boston and Chicago: Ginn & Co.

In preparing this work the aim has been to furnish just so much of Trigonometry as is actually taught in our best schools and colleges. Consequently all investigations that are important only for the special student have been omitted, except the development of functions in series. The principles have been unfolded with the utmost brevity consistent with simplicity and clearness, and interesting problems have been selected with a view of awakening a real love for the study. *Preface.*

The book is a good one and is most admirably adapted to the purpose for which it was prepared. B. F. F.

The Elements of Co-ordinate Geometry. By S. L. Loney, M. A., Late Fellow of Sidney Sussex College, Cambridge, Professor at the Royal Holloway College, and author of a Treatise on Elementary Dynamics, a Treatise on Plane Trigonometry, etc. 8vo. cloth, 416 pp. Price, $1.25. New York: Macmillan & Co.

This excellent book exemplifies the sound judgment and painstaking care which characterizes all of Professor Loney's mathematical works. He is putting himself in the front rank of mathematical writers of the present time, and his books will produce a healthful influence on the mathematicians of the future. We hope that the next book Mr. Loney writes will be a treatise on Spherical Trigonometry, thus making his Treatise on Trigonometry the most complete and admirable Treatise with which we are acquainted.
B. F. F.

The Mathematical Visitor. Edited and published by Artemas Martin, M. A., Ph. D., LL. D., United States Coast Survey, Washington, D. C. Quarto, 18 pp. Price, 50 cents. Issued annually.

The number for 1894 has just reached us. In it is published a number of different solutions of five different problems in Probability. The solutions are by Henry Heaton, G. B. M. Zerr, and the late Professor E. B. Seitz. Five excellent solutions of a difficult problem concerning the *curve of concealment* are also published. The solutions are by Dr. E. A. Bowser, Henry Heaton, Dr. Martin, the late Dr. J. E. Hendricks, and Charles H. Kummell. Two other interesting solutions of a problem are published. These are by J. F. W. Scheffer, and J. A. Pollard.

We regret very much that the ill health of Dr. Martin prevents his publishing the *Mathematical Visitor* and the *Mathematical Magazine* regularly. These two magazines are the type of excellence and beauty. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single number, 10 cents.

For complete and immediate revolution of transportation methods, involving a reduction of freight charges on grain from the West to New York of from 50 to 60 per cent, is what is predicted in the November *Cosmopolitan*. The plan proposes using light and inexpensive corrugated iron cylinders, hung on a slight rail supported on poles from a cross-arm—the whole system involving an expense of not more than fifteen hundred dollars a mile for construction. The rolling stock is equally simple and comparatively inexpensive. Continuous lines of cylinders, moving with no interval to speak of, would carry more grain in a day than a quadruple track railway. This would constitute a sort of grain-pipe line. *The Cosmopolitan* also points out the probable abolition of street cars before the coming horseless carriage, which can be operated by a boy on asphalt pavements at a total expense for labor, oil, and interest, of not more than one dollar a day. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York City.

Foreign affairs naturally have more than usual prominence in the November *Review of Reviews*. In the "Progress of the World," the department of that periodical in which the editor rapidly reviews the events of the preceding month, the possibilities of war in the far East are pointedly set forth. Another theme suggested in the same connection is the progress of Christian missions in the Orient. The prospects of Japan and Russia as Eastern powers are tersely discussed. The editor also comments briefly on the relations of Russia and France, the Italian celebrations, the French victory in Madagascar, the Cuban situation, and British policy in Venezuela. Among home topics of the month, the coming elections, the condition of New York politics, the anti-prize-fight campaign in the Southwest, and the educational outlook are selected for treatment. B. F. F.